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Practice test 1
Practice test 2

Answers and mark schemes can be downloaded for free at www.ntk.edu.hk.
In Paper 1 of the IB SL exam, you are expected to know the properties of the graphs of some basic functions. In Paper 2, you may also need to use a graphic display calculator (GDC) to analyse the graphs of more complicated functions.

**Basic terminology:**

**y-intercept:** The intersection of the graph of \( f(x) \) and the \( y \)-axis

**x-intercept:** The intersection of the graph of \( f(x) \) and the \( x \)-axis

**Asymptote:** A straight line whose distance to the graph of \( f(x) \) tends to zero (that is, the graph of \( f(x) \) gets closer and closer to the asymptote.)

**EXAM TIP**
The \( x \)-intercept and \( y \)-intercept can usually be found by a GDC in Paper 2.

### 5.1 Quadratic functions and their graphs

**General form**

The general expression of a quadratic function is \( f(x) = ax^2 + bx + c \), where \( a \neq 0 \). Its graph is in the shape of a parabola. The orientation of the parabola is determined by the sign of the coefficient \( a \) of the \( x^2 \) term. The lowest or highest point on the parabola is called the vertex (plural: “vertices”).

If \( a > 0 \), the parabola opens upward.

If \( a < 0 \), the parabola opens downward.

- e.g. \( f(x) = x^2 + 2x \)
- e.g. \( f(x) = -x^2 + 2x + 10 \)
From the graphs above, you can see that the graph of a quadratic function is symmetric about a vertical straight line through the vertex. The equation of the line is given by:

\[ x = -\frac{b}{2a} \]

This is given in your formula booklet.

This vertical line is called the **axis of symmetry**.

**Vertex form**

By **completing the square**, we can rewrite the quadratic function \( f(x) = ax^2 + bx + c \) in **vertex form**.

If \( f(x) = ax^2 + bx + c \), then \( f(x) \) can be written as \( f(x) = a(x - h)^2 + k \), where \((h, k)\) are the coordinates of the vertex of the graph of \( f \).

**Example 5-1**

Find the coordinates of the vertex of the graph of

- \( a \) \( y = x^2 + 6x - 1 \)
- \( b \) \( y = x^2 - 3x + 5 \)

**Solution**

- \( a \) \( y = x^2 + 6x - 1 \)
  
  \[ = (x + 3)^2 - 9 - 1 \]
  
  \[ = (x + 3)^2 - 10 \]

  Vertex is at \((-3, -10)\).
b \( y = x^2 - 3x + 5 \)

\[
= \left( x - \frac{3}{2} \right)^2 - \frac{9}{4} + 5
\]

vertex is at \( \left( \frac{3}{2}, \frac{11}{4} \right) \).

If the coefficient of \( x^2 \) is not 1, we have to perform a couple of extra steps to obtain the vertex form:

**Example 5-2**

Find the coordinates of the vertex of the graph of the function \( f(x) = -2x^2 + 12x + 1 \).

**Solution**

\[
f(x) = -2x^2 + 12x + 1
\]

\[
= -2 \left[ x^2 - 6x + \frac{1}{2} \right]
\]

\[
= -2 \left[ (x-3)^2 - 9 + \frac{1}{2} \right]
\]

\[
= -2 \left[ (x-3)^2 - \frac{19}{2} \right]
\]

\[
= -2(x-3)^2 + 19
\]

Vertex is at \( (3, 19) \).
The expression under the radical sign \( b^2 - 4ac \) is called the **discriminant** and is usually denoted by \( \Delta \). Its value enables us to find the number of real solutions to the corresponding quadratic equation.

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>Number of real solutions</th>
<th>Typical graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta = b^2 - 4ac &gt; 0 )</td>
<td>Two distinct real solutions (two ( x )-intercepts)</td>
<td><img src="https://via.placeholder.com/150" alt="Graph" /></td>
</tr>
<tr>
<td>( \Delta = b^2 - 4ac = 0 )</td>
<td>One real solution (one ( x )-intercept)</td>
<td><img src="https://via.placeholder.com/150" alt="Graph" /></td>
</tr>
<tr>
<td>( \Delta = b^2 - 4ac &lt; 0 )</td>
<td>No real solutions (no ( x )-intercepts)</td>
<td><img src="https://via.placeholder.com/150" alt="Graph" /></td>
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**Example 5-8**

How many real solutions do the following equations have?

- \( a \quad x^2 + x + 2 = 0 \)
- \( b \quad -x^2 - 2x + 5 = 0 \)
- \( c \quad 4x^2 + 8x + 4 = 0 \)

**Solution**

- \( a \quad a = 1, \ b = 1, \ c = 2 \)

  Hence \( \Delta = b^2 - 4ac \)
  
  \[ = (1)^2 - 4(1)(2) \]
  
  \[ = 1 - 8 \]
  
  \[ = -7 \]
  
  \[ < 0 \]

  \( \therefore \) no real solutions

---

**COMMON MISTAKE**

Don’t confuse the discriminant and the quadratic formula. The discriminant only tells you how many solutions there are; the quadratic formula gives the actual values.
Step 3 – A vertical stretch by a scale factor of $\frac{1}{3}$: $y = \frac{1}{3}(2x - 4)^2$

Step 4 – A vertical translation of 2 units upward: $y = \frac{1}{3}(2x - 4)^2 + 2$

Warm-up Exercise 5D

The graph of $y = f(x)$ is shown below.

Sketch the graph of each of the following:

a. $y = f(x) + 3$

b. $y = f(x + 3)$

c. $y = -f(x)$

d. $y = f(x - 3)$

e. $y = f(2x)$

f. $y = 2f(x)$
The graph of \( y = f(x) \) is shown below.

Sketch the graph of each of the following:

\[
\begin{align*}
\text{a} & \quad y = f(x) + 2 \\
\text{b} & \quad y = f(x - 2) \\
\text{c} & \quad y = f(-x) \\
\text{d} & \quad y = -f(x + 3) \\
\text{e} & \quad y = f(2x) \\
\text{f} & \quad y = 2f(x)
\end{align*}
\]

The graph of \( y = f(x) \) is shown below:

Sketch the graph of the following functions.

\[
\begin{align*}
\text{a} & \quad y = f(x) - 1 \\
\text{b} & \quad y = f(x + 2) \\
\text{c} & \quad y = f(-x) \\
\text{d} & \quad y = -f(x) \\
\text{e} & \quad y = f(2x) \\
\text{f} & \quad y = \frac{1}{2}f(x)
\end{align*}
\]

Describe exactly the two transformations required to obtain the graph of each of the following from the graph of \( y = f(x) \).

\[
\begin{align*}
\text{a} & \quad y = 2f(x) + 4 \\
\text{b} & \quad y = 2f(2x) \\
\text{c} & \quad y = f(-x) + 3 \\
\text{d} & \quad y = -f(x + 2)
\end{align*}
\]
Express the function \( g(x) \) in terms of \( f(x) \) where the graph of \( y = g(x) \) can be obtained by applying the following transformations to the graph of \( y = f(x) \).

a. Translate by the vector \( \begin{pmatrix} -2 \\ 3 \end{pmatrix} \).
b. Move downward 2 units and then move to the right by 3 units.
c. Move to the left 3 units and then stretch vertically by a factor of \( \frac{1}{2} \).
d. Reflect about the \( x \)-axis and then stretch horizontally by a factor of \( \frac{1}{5} \).
e. Stretch horizontally by a factor of 3 and then stretch vertically by a factor of \( \frac{1}{3} \).
f. Reflect about the \( y \)-axis and then move downward by 2 units.
g. Move upward by 7 units and then reflect about the \( x \)-axis.

---

**Exam Practice 5D**

**Paper 1**

1. The graph of \( y = f(x) \), \(-3 \leq x \leq 3\), is shown below. Sketch the graph of \( y = f(x-1)+2 \) on the same axes.

2. The graph of \( y = f(x) \) is transformed into the graph of \( y = 5f(x-3)+1 \). Give a full geometric description of this transformation.
3 The graph of \( y = f(x), \ -3 \leq x \leq 3 \), is shown below. Sketch the graph of 
\[ y = -f(2x) \] on the same axes.

![Graph of \( y = f(x) \) and \( y = -f(2x) \)]

4 The graph of \( y = f(x), \ -4 \leq x \leq 4 \), is shown below. The point \( A(3,1) \) lies on the graph, \( x = 2 \) is a vertical asymptote, and \( y = 0 \) is a horizontal asymptote.

a Write down the equation of the new \( x \)-asymptote if \( f(x) \) is translated 3 units to the left.

b Sketch the graph of \( y = f(x+3) - 2 \) on the same axes.

c The point \( A \) on the graph of \( f \) is mapped to the point \( A' \) on the graph of 
\[ y = f(x + 3) - 2 \]. Find the coordinates of \( A' \).

5 The graph of \( y = f(x), \ -4 \leq x \leq 4 \), is shown below. Draw the graph of 
\[ y = -f(x) - 2 \] on the same axes. Mark \( A' \) and \( B' \), the image of \( A \) and \( B \) respectively, on your graph, together with their coordinates.

![Graph of \( y = f(x) \) and \( y = -f(x) - 2 \)]
Summary

Quadratic functions

The graph of a quadratic function looks like this:

\[ y = f(x) \]

\[ x = \frac{-b}{2a} \text{ or } x = \frac{(p+q)}{2} \]

We can express the quadratic function in:

1. **General form** \( f(x) = ax^2 + bx + c \)
   - The parabola opens upward if \( a > 0 \) and the parabola opens downward if \( a < 0 \). The \( y \)-intercept is \( c \).
   - The axis of symmetry can be found by \( x = \frac{-b}{2a} \).

2. **Vertex form (completed square form)** \( f(x) = a(x - h)^2 + k \)
   - The point \( (h, k) \) is the vertex of the graph of \( f \).
   - The axis of symmetry is \( x = h \).

3. **Intercept form** \( f(x) = a(x - p)(x - q) \)
   - The \( x \)-intercepts of the graph of \( f \) are \( p \) and \( q \).
   - The axis of symmetry can be found using \( x = \frac{p + q}{2} \).

The number of real roots can be determined by the discriminant \( \Delta = b^2 - 4ac \).
- If \( \Delta > 0 \) \( \Rightarrow \) two distinct real roots.
- If \( \Delta = 0 \) \( \Rightarrow \) one root (repeated root, equal roots).
- If \( \Delta < 0 \) \( \Rightarrow \) no real roots.

Inverse functions

The graph of the inverse function \( f^{-1}(x) \) is the reflection of the graph \( f(x) \) about the line \( y = x \).