Chapter 1 — Exponents and logarithms

Conversion between exponents and logarithms

The conversion between exponents and logarithms is defined below:

For $a > 0$, $a^x = b \Leftrightarrow x = \log_a b$.

When $a = 10$, we express the operator $\log_{10}$ as $\log$ and this can be found on the GDC.

e is a special constant with numerical value $2.71828\ldots$ and can be found on the GDC.

When $a = e$, we can express the operator $\log_e$ as $\ln$ and this can be found on the GDC also.

Example: by using the above notation, we have the following equivalent equations:

$4^3 = 64 \Leftrightarrow 3 = \log_4 64$

$10^4 = 10000 \Leftrightarrow 4 = \log 10000$

$e^2 = 7.39$ (3sf) $\Leftrightarrow 2 = \ln 7.39$

Laws of exponents

The following table lists the laws of exponents (m and n are real numbers while a and b are non-zero real numbers unless specified otherwise.)

Laws	Examples
$a^m a^n = a^{m+n}$	$(3^4)(3^2) = 3^{4+2} = 3^6$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{2^7}{2^3} = 2^{7-3} = 2^4$
$(a^m)^n = a^{mn}$	$(5^3)^4 = 5^{3\times4} = 5^{12}$
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m,\ n \neq 0$	$16^{\frac{3}{2}} = \left(\sqrt{16}\right)^3 = 4^3 = 64,\ 27^{\frac{5}{3}} = \left(\sqrt[3]{27}\right)^5 = 3^5 = 243$
$(ab)^n = a^n b^n$	$(2x)^7 = 2^7 x^7 = 128x^7$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{k}{3}\right)^4 = \frac{k^4}{3^4} = \frac{k^4}{81}$
$a^{-n} = \frac{1}{a^n}$	$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\left(\frac{7}{9}\right)^{-3} = \left(\frac{9}{7}\right)^3 = \frac{9^3}{7^3}$
$a^0 = 1,\ a \neq 0$	$68^0 = 1$
$0^n = 0,\ n > 0$	$0^{295} = 0$

There are two points to note here:

(i) 0 to the power of a negative number does not work because, for instance, $0^{-5} = \frac{1}{0^5} = \frac{1}{0}$ and we cannot divide any number by 0.

(ii) A negative number to an even power will get a positive answer, but a negative number to an odd power will get a negative answer. There is another way to remember this:

$$\begin{cases} \text{even power of a non-zero number is always positive} \\ \text{odd power of a non-zero number is always the same sign as before} \end{cases}$$

Laws of logarithms

Laws	Examples
$\log_b x + \log_b y = \log_b xy$	$\log_7 2 + \log_7 5 = \log_7(2(5)) = \log_7 10$
$\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$	$\log_8 13 - \log_8 4 = \log_8\left(\frac{13}{4}\right)$
$\log_b(x^p) = p\log_b x$	$\log_{11}(6^5) = 5\log_{11}6$
$\log_a a = 1$	$\log_5 5 = 1$
$\log_a 1 = 0$	$\log_{31} 1 = 0$
$a^x = e^{x\ln a}$	$9^7 = e^{7\ln 9}$
$\log_a a^x = x = a^{\log_a x}$	$\log_6 6^8 = 8 = 6^{\log_6 8}$
$\log_b a = \frac{\log_c a}{\log_c b}$	$\log_4 11 = \frac{\log_3 11}{\log_3 4}$

The last one is called the change of base formula.

To input a logarithmic function into the GDC, we use the change of base formula to convert the base of the logarithmic function to either 10 or e so that we can use $\log$ or $\ln$ in the GDC. Therefore, we have $\log_b a = \frac{\log_{10} a}{\log_{10} b} = \frac{\log a}{\log b}$ and $\log_b a = \frac{\log_e a}{\log_e b} = \frac{\ln a}{\ln b}$.

Note that $\log_a x$ is undefined if $x \leq 0$. So $\log_2(0)$, $\log_4(-3)$ and $\ln(-2.5)$ are undefined. We'll investigate more in chapter 8.

Example: for $\log_3 72$, we can express it in terms of $\log_3 2$ as follows:

$\log_3 72$

$= \log_3(2^3 \times 3^2)$

$= \log_3 2^3 + \log_3 3^2$

$= 3\log_3 2 + 2\log_3 3$

$= 3\log_3 2 + 2(1)$

$= 3\log_3 2 + 2$

Example: to find the solution of the equation $7^x = 19$ to 3 significant figures, we need to make use of the change of base formula and the GDC:

$7^x = 19$

$\Rightarrow x = \log_7 19 = \dfrac{\log 19}{\log 7} = 1.51$ (3sf)

Worked examples

Example question 1.1: (HL/SL Paper 1 Section A)

Find the lowest common multiple of $30x^2y^3z^5$ and $36x^3yz^4$.

Solution:

Since we can express 30 as $2(3)(5)$ and 36 as $2^2 3^2$, the lowest common multiple of $30x^2y^3z^5$and $36x^3yz^4$ is $2^2 3^2 5x^3y^3z^5 = 180x^3y^3z^5$.

Example question 1.2: (HL/SL Paper 1 Section A)

Solve $\log_3\left((x-3)^2\right) = 4$ for x.

Solution:

$\log_3(x-3)^2 = 4$

$\Rightarrow (x-3)^2 = 3^4$

$\Rightarrow (x-3)^2 = 81$

$\Rightarrow x - 3 = \pm 9$

$\Rightarrow x = 12$ or -6

Example question 1.3: (HL/SL Paper 2 Section A)

Solve $(7x)^{-\frac{3}{2}}\left(\frac{14}{3}\right)\left(\frac{12}{x^2}\right)^2 = 4x^{-5}$.

Solution:

$$(7x)^{-\frac{3}{2}}\left(\frac{14}{3}\right)\left(\frac{12}{x^2}\right)^2 = 4x^{-5}$$

$$\Rightarrow \frac{1}{7^{\frac{3}{2}}x^{\frac{3}{2}}}\left(\frac{2(7)}{3}\right)\left(\frac{144}{x^4}\right) = \frac{4}{x^5}$$

$$\Rightarrow \frac{24}{7^{\frac{1}{2}}} = x^{\frac{1}{2}}$$

$$\Rightarrow x = \frac{576}{7}$$

Example question 1.4: (HL/SL Paper 2 Section A)

Solve $\log_x 13 + 2 = \ln 17$.

Solution:

$$\log_x 13 + 2 = \ln 17$$

$$\Rightarrow \log_x 13 = \ln 17 - 2$$

$$\Rightarrow \frac{\log 13}{\log x} = \ln 17 - 2$$

$$\Rightarrow \log x = \frac{\log 13}{\ln 17 - 2}$$

$$\Rightarrow x = 10^{\left(\frac{\log 13}{\ln 17 - 2}\right)} = 21.7 \text{ (3sf)}$$

Exercises

Exercise 1.1 (HL/SL Paper 1 Section A)

Simplify $\left(\frac{4}{9}x^2\right)^{-\frac{1}{2}}\left(\frac{-\frac{7}{3}}{(27x)^{-\frac{1}{3}}}\right)$ and give the answer in terms of a positive power of x.

Answer: $-\frac{21}{2x^{\frac{2}{3}}}$

Exercise 1.2 (HL/SL Paper 1 Section A)

Express $\log_2\left(\frac{25}{16}\right)$ in terms of k where $k = \log_2 5$.

Answer: $2k - 4$

Exercise 1.3 (HL/SL Paper 2 Section A)

Find the highest common factor of $56875a^2b^5c^7$ and $44200a^{10}b^4c^3$.

Answer: $325a^2b^4c^3$

Exercise 1.4 (HL/SL Paper 2 Section A)

Solve $2(13^x) = (5e)^{-x}$. Give the answer to 3 significant figures.

Answer: -0.134